EXERCISE SOLUTIONS, LECTURES 1-7

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1. Overview, coordinate systems

Exercise 1. Find the rectangular coordinates (in 2D) of the point $(r, \theta) = (2, \frac{\pi}{4})$ in polar coordinates.

Solution. We have

$$x = r\cos\theta = 2\cos\frac{\pi}{4} = \sqrt{2},$$
$$y = r\sin\theta = 2\sin\frac{\pi}{4} = \sqrt{2}.$$

□

Exercise 2. Find the polar coordinates of the point $(x, y) = (1, -1)$ in rectangular coordinates (in 2D).

Solution. If you plot the point you realize that θ must be $-45°$, or $-\frac{\pi}{4}$ (or if you prefer $\frac{7\pi}{4}$). Indeed

$$\cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}},$$
$$\sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}},$$

so $r = \sqrt{2}$ and $\theta = \frac{7\pi}{4}$ will do. □

Exercise 3. Find the rectangular coordinates (in 3D) of the point $(r, \theta, z) = (4, \frac{4\pi}{3}, 1)$ in cylindrical coordinates.

Solution. We have

$$x = r\cos\theta = 4\cos\frac{4\pi}{3} = -2,$$
$$y = r\sin\theta = 4\sin\frac{4\pi}{3} = -2\sqrt{3},$$
$$z = 1.$$

□

Exercise 4. Find the cylindrical coordinates of the point $(x, y, z) = (0, -1, 3)$ in rectangular coordinates (in 3D).

Solution. If you plot the point you realize that $\theta = 3\pi/2$. And $r = \sqrt{x^2 + y^2} = 1$, and $z = 3$. ☐

Exercise 5. Explain why $\rho = \sqrt{r^2 + z^2}$ (in the context of expressing spherical coordinates in terms of cylindrical coordinates).

Solution. Using rectangular coordinates, we have $r^2 = x^2 + y^2$, and $\rho = \sqrt{x^2 + y^2 + z^2}$. So we have $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$. ☐

Exercise 6. Convert the point $(x, y, z) = (0, -2, 0)$ in rectangular coordinates (in 3D) to spherical coordinates.

Solution. Since the point is on the xy-plane, $\phi = \frac{\pi}{2}$. Because it is on the negative y-axis, $\theta = 3\pi/2$. Then $\rho = \sqrt{x^2 + y^2 + z^2} = 2$. ☐

Exercise 7. Convert the point $(\rho, \theta, \phi) = (2, \frac{\pi}{4}, \frac{\pi}{4})$ to rectangular coordinates (in 3D).

Solution. We have

$$\begin{aligned} x &= \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{4} \cos\frac{\pi}{4} = 1, \\ y &= \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{4} \sin\frac{\pi}{4} = 1, \\ z &= \rho \cos\phi = 2 \cos\frac{\pi}{4} = \sqrt{2}. \end{aligned}$$

☐

Exercise 8. Express the equation $\phi = \frac{\pi}{4}$, in spherical coordinates, of the cone in terms of cylindrical coordinates.

Solution. We have

$$\tan\phi = \frac{r}{z},$$

so in cylindrical coordinates we have $r = z$. ☐

Exercise 9. Consider a vertical line (in 2D) through the point $(3, 3)$ (expressed in rectangular coordinates). Decide if the line would be more easily expressed in polar coordinates or in rectangular coordinates. Then write an equation for it.

Solution. In rectangular coordinates, this is just $x = 3$. In polar coordinates, this is $r\cos\theta = 3$. So I would say it is easier in rectangular coordinates. ☐

Exercise 10. Find an equation in polar coordinates for the curve represented by the given equation in rectangular coordinates (in 2D).

- $x^2 + y^2 = 7$.
- $x = -1$.
- $y = \sqrt{3}x$.
- $x^2 + y^2 = 4y$.

Solution.

- Since $r^2 = x^2 + y^2$, we have $r^2 = 7$, or $r = \sqrt{7}$.
- Since $x = r\cos\theta$, we have $r\cos\theta = -1$.

- Since $\tan\theta = \frac{y}{x}$, we have $\tan\theta = \sqrt{3}$. This means $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$.
- Since $r^2 = x^2 + y^2$ and $y = r\sin\theta$, we have

$$r^2 = 4r\sin\theta,$$

or

$$r = 4\sin\theta.$$

□

2. Vectors

Exercise 1. Is each of the following quantities a vector or a scalar? Explain.

(1) The current temperature
(2) The population of the world
(3) The current in a river

Solution.

(1) It is a numerical value, so it is a scalar.
(2) It's a number, so it's a scalar.
(3) This contains the information on the direction, so this is a vector.

□

Exercise 2. What is the relationship between the point $(4, 7)$ and the vector $\langle 4, 7\rangle$?

Solution. The vector $\langle 4, 7\rangle$ is the vector formed by the arrow whose starting point is the origin and the endpoint is the point $(4, 7)$. □

Exercise 3. If $\vec{u} = \langle -3, 4\rangle$ and $\vec{v} = \langle 9, -1\rangle$, find $\vec{u} + 3\vec{v}$, $2\vec{u}$ and $|\vec{u} - \vec{v}|$.

Solution. We have

$$\vec{u} + 3\vec{v} = \langle -3, 4\rangle + 3\langle 9, -1\rangle = \langle -3 + 27, 4 - 3\rangle = \langle 24, 1\rangle,$$
$$2\vec{u} = 2\langle -3, 4\rangle = \langle -6, 8\rangle,$$
$$|\vec{u} - \vec{v}| = |\langle -3, 4\rangle - \langle 9, -1\rangle| = |\langle -12, 5\rangle| = \sqrt{(-12)^2 + 5^2} = \sqrt{169} = 13.$$

□

Exercise 4. Find a unit vector that has the same direction as $8\vec{i} + \vec{j} - 4\vec{k}$.

Solution. Since

$$|8\vec{i} + \vec{j} - 4\vec{k}| = \sqrt{8^2 + 1^2 + (-4)^2} = \sqrt{64 + 1 + 16} = \sqrt{81} = 9,$$

the unit vector in the same direction is

$$\frac{1}{9}(8\vec{i} + \vec{j} - 4\vec{k}) = \langle \frac{8}{9}, \frac{1}{9}, -\frac{4}{9}\rangle.$$

□

Exercise 5. Find a vector represented by the arrow $\overrightarrow{AB}$, for $A = (-2, 1)$ and $B = (5, 5)$.

Solution.

$$\overrightarrow{AB} = \langle 5 - (-2), 5 - 1\rangle = \langle 7, 4\rangle.$$

□

Exercise 6. If $\vec{u} = \langle 3, 1, -4\rangle$ and $\vec{v} = \langle 0, 1, -1\rangle$, find $\vec{u} - 3\vec{v}$ and $-\vec{u} - \vec{v}$.

Solution. We have

$$\vec{u} - 3\vec{v} = \langle 3, 1, -4\rangle - 3\langle 0, 1, -1\rangle = \langle 3-0, 1-3, -4+3\rangle = \langle 3, -2, -1\rangle,$$
$$-\vec{u} - \vec{v} = -\langle 3, 1, -4\rangle - \langle 0, 1, -1\rangle = \langle -3-0, -1-1, 4+1\rangle = \langle -3, -2, 5\rangle.$$

□

3. Dot product

Exercise 1. What is the angle between $\vec{i} + \sqrt{3}\vec{j}$ and the positive x-direction?

Solution. The positive x-direction is represented by a vector $\langle 1, 0\rangle$, so the angle θ satisfies

$$\cos\theta = \frac{\langle 1, \sqrt{3}\rangle \cdot \langle 1, 0\rangle}{|\langle 1, \sqrt{3}\rangle|\,|\langle 1, 0\rangle|} = \frac{1}{\sqrt{1^2 + \sqrt{3}^2}\sqrt{1^2 + 0^2}} = \frac{1}{2},$$

or $\theta = \frac{\pi}{3}$. □

Exercise 2. Find $\vec{u} \cdot \vec{v}$ for the following.

(1) $\vec{u} = \langle 1, 3, -5\rangle, \vec{v} = \langle 4, 3, 19\rangle$
(2) $\vec{u} = \langle 0, 4\rangle, \vec{v} = \langle 8, -6\rangle$
(3) $\vec{u} = \langle 7, 11\rangle, \vec{v} = \langle 9, 2\rangle$
(4) $\vec{u} = 2\vec{i} + \vec{k}, \vec{v} = \vec{i} - 6\vec{j}$

Solution.

(1) $\vec{u} \cdot \vec{v} = 1 \cdot 4 + 3 \cdot 3 + (-5) \cdot 19 = 4 + 9 - 95 = -82.$
(2) $\vec{u} \cdot \vec{v} = 0 \cdot 8 + 4 \cdot (-6) = -24.$
(3) $\vec{u} \cdot \vec{v} = 7 \cdot 9 + 11 \cdot 2 = 63 + 22 = 85.$
(4) $\vec{u} \cdot \vec{v} = \langle 2, 0, 1\rangle \cdot \langle 1, -6, 0\rangle = 2.$

□

Exercise 3. Show that $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$ and $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$.

Solution. For example, $\langle 1, 0, 0\rangle \cdot \langle 0, 1, 0\rangle = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$, etc. □

Exercise 4. Let θ be the angle between $\vec{u} = \langle 5, 1\rangle$ and $\vec{v} = \langle 3, 2\rangle$. What is $\cos\theta$?

Solution. The angle θ satisfies

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{15 + 2}{\sqrt{5^2 + 1^2}\sqrt{3^2 + 2^2}} = \frac{17}{\sqrt{26}\sqrt{13}} = \frac{17}{13\sqrt{2}}.$$

□

Exercise 5. Let θ be the angle between $\vec{u} = \vec{i} - 4\vec{j} + \vec{k}$ and $\vec{v} = -3\vec{i} + \vec{j} + 5\vec{k}$. What is $\cos\theta$?

Solution. The angle θ satisfies

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{\langle 1, -4, 1\rangle \cdot \langle -3, 1, 5\rangle}{\sqrt{1^2 + (-4)^2 + 1^2}\sqrt{(-3)^2 + 1^2 + 5^2}} = \frac{-3 - 4 + 5}{\sqrt{18}\sqrt{35}} = -\frac{2}{\sqrt{630}}.$$

□

Exercise 6. Determine whether the triangle with vertices $P = (1, -3, -2)$, $Q = (2, 0, -4)$, $R = (6, -2, -5)$ is right-angled.

Solution. If the triangle has a right angle at P, then $\overrightarrow{PQ} = \langle 1, 3, -2\rangle$ and $\overrightarrow{PR} = \langle 5, 1, -3\rangle$ should be orthogonal to each other. Since $\overrightarrow{PQ} \cdot \overrightarrow{PR} = 5 + 3 + 6 \neq 0$, the angle at P is not right-angled.

If the triangle has a right angle at Q, then $\overrightarrow{QP} = \langle -1, -3, 2\rangle$ and $\overrightarrow{QR} = \langle 4, -2, -1\rangle$ should be orthogonal to each other. Since $\overrightarrow{QP} \cdot \overrightarrow{QR} = -4 + 6 - 2 = 0$, the angle at Q is a right angle! So the triangle is right-angled. □

Exercise 7. Find the values of x such that the angle between the vectors $\langle 2, 1, -1\rangle$ and $\langle 1, x, 0\rangle$ is $\frac{\pi}{4} = 45°$.

Solution. This means that

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \frac{\langle 2, 1, -1\rangle \cdot \langle 1, x, 0\rangle}{\sqrt{2^2 + 1^2 + (-1)^2}\sqrt{1^2 + x^2 + 0^2}} = \frac{2 + x}{\sqrt{6}\sqrt{x^2 + 1}}.$$

So

$$\frac{1}{2} = \frac{(x+2)^2}{6x^2 + 6} = \frac{x^2 + 4x + 4}{6x^2 + 6},$$

or

$$3x^2 + 3 = x^2 + 4x + 4,$$

or

$$2x^2 - 4x - 1 = 0.$$

So

$$x = \frac{4 \pm \sqrt{16 + 8}}{4} = \frac{4 \pm 2\sqrt{6}}{4} = \frac{2 \pm \sqrt{6}}{2}.$$

Note that we have to plug back into the original equation because we've squared along the way, so there might be a sign issue. Since

$$2 + x = \frac{6 \pm \sqrt{6}}{2},$$

it is always positive, so both values of $x = \frac{2\pm\sqrt{6}}{2}$ are possible. □

Exercise 8. Find the $\vec{v}$-direction component of $\vec{u}$, where $\vec{u} = \langle 3, -1, 1\rangle$ and $\vec{v} = \langle 4, 7, -4\rangle$.

Solution. This is

$$\operatorname{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{3 \cdot 4 + (-1) \cdot 7 + 1 \cdot (-4)}{\sqrt{4^2 + 7^2 + (-4)^2}} = \frac{12 - 7 - 4}{\sqrt{16 + 49 + 16}} = \frac{1}{\sqrt{81}} = \frac{1}{9}.$$

□

Exercise 9. Find the projection of $\vec{u} = 5\vec{j} - \vec{k}$ to $\vec{v} = 2\vec{i} + \vec{j} + 3\vec{k}$.

Solution. This is

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} = \frac{5 - 3}{2^2 + 1^2 + 3^2}\langle 2, 1, 3\rangle = \frac{2}{14}\langle 2, 1, 3\rangle = \langle \frac{2}{7}, \frac{1}{7}, \frac{3}{7}\rangle.$$

□

Exercise 10. Show that, for any vectors $\vec{u}$ and $\vec{v}$ (in either 2D or 3D), $\vec{v} - \operatorname{proj}_{\vec{u}} \vec{v}$ is orthogonal to $\vec{u}$.

Solution. We want to show $(\vec{v} - \operatorname{proj}_{\vec{u}} \vec{v}) \cdot \vec{u} = 0$. Note

$$(\vec{v} - \operatorname{proj}_{\vec{u}} \vec{v}) \cdot \vec{u} = \vec{v} \cdot \vec{u} - (\operatorname{proj}_{\vec{u}} \vec{v}) \cdot \vec{u},$$

so we need to show $(\operatorname{proj}_{\vec{u}} \vec{v}) \cdot \vec{u} = \vec{u} \cdot \vec{v}$. Note

$$(\operatorname{proj}_{\vec{u}} \vec{v}) \cdot \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u} \cdot \vec{u} = \vec{u} \cdot \vec{v},$$

which is what we wanted. □

4. Cross product

Exercise 1. Find $\vec{u} \times \vec{v}$:

(1) $\vec{u} = \langle 0, 5, -4 \rangle$, $\vec{v} = \langle 1, 2, -1 \rangle$
(2) $\vec{u} = \langle 5, 6, 2 \rangle$, $\vec{v} = \langle 0, -4, -3 \rangle$
(3) $\vec{u} = \langle -1, -5, -3 \rangle$, $\vec{v} = \langle 4, -1, -3 \rangle$
(4) $\vec{u} = \langle 100, 200, 300 \rangle$, $\vec{v} = \langle -1, -2, -3 \rangle$

Solution.

(1) $\langle 3, -4, -5 \rangle$
(2) $\langle -10, 15, -20 \rangle$
(3) $\langle 12, -15, 21 \rangle$
(4) $\langle 0, 0, 0 \rangle$

□

Exercise 2. True or False:

(1) If $\vec{u} \times \vec{v} = \vec{0}$, then $\vec{u}$ and $\vec{v}$ are parallel to each other.
(2) $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.
(3) If $\vec{u} \times \vec{v} = \vec{0}$ and $\vec{u} \times \vec{w} = \vec{0}$, then $\vec{v} \times \vec{w} = \vec{0}$.
(4) If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$, then $\vec{u}$ is either parallel to $\vec{v}$ or parallel to $\vec{w}$.

Solution.

(1) True. This is because $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$, and this is zero if and only if either $|\vec{u}| = 0$, $|\vec{v}| = 0$ or $\sin\theta = 0$. If either $|\vec{u}| = 0$ or $|\vec{v}| = 0$, this means either $\vec{u}$ or $\vec{v}$ is $\vec{0}$, and $\vec{0}$ is parallel to any vector. If $\sin\theta = 0$, then $\theta = 0$ or π, and both cases are also the parallel cases.
(2) True. This is because $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ (and to $\vec{v}$).
(3) False. For example, $\vec{u} = \vec{0}$, $\vec{v} = \vec{i}$, $\vec{w} = \vec{j}$. Then $\vec{u} \times \vec{v} = \vec{0} \times \vec{i} = \vec{0}$ and $\vec{u} \times \vec{w} = \vec{0} \times \vec{j} = \vec{0}$, but $\vec{v} \times \vec{w} = \vec{i} \times \vec{j} = \vec{k}$.
(4) False. If $\vec{v} = \vec{i}$ and $\vec{w} = \vec{j}$ and $\vec{u} = \vec{i} + \vec{j}$, then $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot \vec{k} = 0$, but $\vec{u}$ is parallel to neither $\vec{v}$ nor $\vec{w}$.

□

Exercise 3. Show that $|\vec{u} \times \vec{v}|^2 + |\vec{u} \cdot \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2$.

Solution. This is because $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$ and $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$, so

$$|\vec{u} \times \vec{v}|^2 + |\vec{u} \cdot \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 (\sin^2\theta + \cos^2\theta) = |\vec{u}|^2 |\vec{v}|^2.$$

□

Exercise 4. Explain why $|\vec{u} \times \vec{v}|$ is the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.

Solution. Note $|\vec{u} \times \vec{v}| = |\vec{u}|\ |\vec{v}| \sin\theta$. We know the area of a parallelogram is the base times the height. If you take the line parallel to $\vec{u}$ as the base, then the height is precisely $|\vec{v}| \sin\theta$. □

Exercise 5. Compute the volume of the parallelepiped formed by $\vec{u} = \langle 2, -5, 0\rangle$, $\vec{v} = \langle -3, 0, 0\rangle$, $\vec{w} = \langle 0, 1, 4\rangle$.

Solution. Using $|\vec{u} \cdot (\vec{v} \times \vec{w})|$, we have $\vec{v} \times \vec{w} = \langle 0, 12, -3\rangle$, and $\vec{u} \cdot (\vec{v} \times \vec{w}) = -60$, so the volume is 60. □

5. Lines and curves

Exercise 1. Find the parametric equations for the following implicit equations (in 2D). Use the parameter t.

(1) $(x-5)^2 + (y-4)^2 = 16$.
(2) $x + 2y = 1$.
(3) $2x^2 + y^2 = 3$.

Solution.

(1) $x = 4\cos t + 5, y = 4 + 4\sin t$.
(2) $x = 1 - 2t, y = t$.
(3) $x = \frac{\sqrt{3}}{\sqrt{2}} \cos t, y = \sqrt{3}\sin t$.

□

Exercise 2. Find the implicit equations for the following parametric equations (in 2D).

(1) $\langle x, y\rangle = \langle \cos t + \sin t, \cos t - \sin t\rangle$
(2) $\langle x, y\rangle = \langle \cos t + 2\sin t + 1, \sin t + 2\rangle$

Solution.

(1) We want to express $\cos t, \sin t$ in terms of x, y. Note that $x+y = 2\cos t$ and $x-y = 2\sin t$. Thus $(x+y)^2 + (x-y)^2 = 4$.
(2) We want to express $\cos t, \sin t$ in terms of x, y. Note that $x - 2y = \cos t - 3$, so $\cos t = x - 2y + 3$, whereas $y = \sin t + 2$, so $\sin t = y - 2$. Since we have $\cos^2 t + \sin^2 t = 1$, we have
$$(x - 2y + 3)^2 + (y-2)^2 = \cos^2 t + \sin^2 t = 1.$$

□

Exercise 3. Find the parametric and equations for the following lines (in 3D).

(1) The line through the points $(1, 2, 6)$ and $(2, 4, 8)$.
(2) The line through the points $(2, 3, 1)$ and $(1, -3, -6)$.

Solution.

(1) The line has the direction vector $\langle 1, 2, 2\rangle$ and passes through $(1, 2, 6)$, so we can express it as
$$x = 1 + t, y = 2 + 2t, z = 6 + 2t.$$
(2) The line has the direction vector $\langle -1, -6, -7\rangle$ and passes through $(2, 3, 1)$, so we can express it as
$$x = 2 - t, y = 3 - 6t, z = 1 - 7t.$$

□

Exercise 4. Find the angle between the line through $(-2,4,0)$ and $(1,1,1)$ and the line through $(2,3,4)$ and $(2,-1,-8)$.

Solution. The first line has the direction vector $\vec{u} = \langle 3,-3,1\rangle$, and the second line has the direction vector $\vec{v} = \langle 0,-4,-12\rangle$. The angle, denoted θ, satisfies

$$|\cos\theta| = \frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|\,|\vec{v}|}.$$

Note $\vec{u}\cdot\vec{v} = 12-12 = 0$, so $\cos\theta = 0$, or $\theta = \frac{\pi}{2}$. □

Exercise 5. Determine whether the following pairs of lines (in 3D) are parallel, intersecting or skew. If they intersect, find the point of intersection.

(1) $L_1 : x = 2-3t, y = 3+2t, z = t$
$L_2 : x = -1+s, y = 5+7s, z = 1-6s$
(2) $L_1 : x = 5+t, y = 2+t, z = -t-1$
$L_2 : x = s-1, y = s-2, z = 6-s$
(3) $L_1 : x = t, y = 1, z = -t-1$
$L_2 : x = 4-s, y = s+1, z = s+3$
(4) $L_1 : x = 4-5t, y = -t+1, z = t+1$
$L_2 : x = 2s+1, y = s-1, z = s+1$

Solution.

(1) They share $(-1,5,1)$ ($t=1$ and $t=0$ work for L_1 and L_2, respectively) and they are not parallel (the direction vectors $\langle -3,2,1\rangle$ and $\langle 1,7,-6\rangle$ are not parallel). so they intersect.
(2) They have the same direction vector, $\langle 1,1,-1\rangle$, so they are parallel.
(3) If they intersect, there should be t, s such that

$$t = 4-s, \quad 1 = s+1, \quad -t-1 = s+3.$$

So $s=0$, and $t = 4-0 = 4$. Then the third equation reads $-4-1 = 0+3$, which does not hold. So, they are skew.
(4) If they intersect, there should be t, s such that

$$4-5t = 2s+1, \quad -t+1 = s-1, \quad t+1 = s+1.$$

Adding the last two equations together, we get $2 = 2s$, so $s = 1$. So $t = s = 1$. But the first equation reads $4-5 = 2+1$, which does not hold. So, they are skew.

□

Exercise 6. True or False:

(1) In 2D, two lines orthogonal to a third line are parallel.
(2) In 3D, two lines orthogonal to a third line are parallel.
(3) In 2D, two lines orthogonal to a third line are orthogonal.
(4) In 3D, two lines orthogonal to a third line are orthogonal.
(5) In 2D, two lines either intersect or are parellel.
(6) In 3D, two lines either intersect or are parellel.
(7) In 2D, two lines parallel to a third line are parallel.
(8) In 3D, two lines parallel to a third line are parallel.

Solution.

(1) True.
(2) False. Example: the vectors $\vec{i}+\vec{j}$ and $\vec{i}$ are orthogonal to the vector $\vec{k}$, but $\vec{i}+\vec{j}$ and $\vec{i}$ are not parallel.
(3) False. They are parallel.
(4) False. The same example as in (2) works.
(5) True.
(6) False. They can be skew.
(7) True.
(8) True.

□

6. Planes and surfaces

Exercise 1. True or False:

(1) Two planes parallel to a third plane are parallel.
(2) Two planes orthogonal to a third plane are parallel.
(3) Two planes parallel to a line are parallel.
(4) Two planes either intersect or are parallel.
(5) A plane and a line either intersect or are parallel.
(6) Two planes orthogonal to a line are parallel.
(7) Two lines parallel to a plane are parallel.
(8) Two lines orthogonal to a plane are parallel.

Solution.

(1) True.
(2) False. Example: $x=0$ and $y=0$ are orthogonal to $z=0$.
(3) False. Example: $x=0$ and $y=0$ are parallel to $x=0, y=0, z=t$.
(4) True.
(5) True.
(6) True.
(7) False. Example: $x=0$ is parallel to $x=0, y=t, z=0$ and $x=0, y=0, z=t$.
(8) True.

□

Exercise 2. Find an equation of the following planes.

(1) The plane that passes through the point $A=(1,0,4)$ and is orthogonal to the vector $\vec{n}=\langle 3,2,-1\rangle$.
(2) The plane that passes through the point $A=(-5,2,1)$ and is parallel to the vectors $\vec{u}=\langle 1,-2,2\rangle$ and $\vec{v}=\langle 4,-1,-2\rangle$.
(3) The plane that passes through the points $A=(2,4,5)$, $B=(1,0,3)$ and $C=(5,8,3)$.
(4) The plane that passes through the points $A=(2,0,-4)$, $B=(0,1,4)$ and $C=(5,2,5)$.

Solution.

(1) It is of the form $3x+2y-z=a$ for some number a such that $(1,0,4)$ is on it. Then $a=3\cdot 1+2\cdot 0-4=-1$. So $3x+2y-z=-1$.

(2) Note that $\vec{u}\times\vec{v} = \langle 6, 10, 7\rangle$ is a normal vector, so it can be expressed as $6x+10y+7z = a$, where a is such that $6\cdot(-5)+10\cdot 2+7\cdot 1 = a$. So $a = -30+20+7 = -3$, and the plane can be expressed as $6x+10y+7z = -3$.

(3) This is the same as the plane that passes through $(2, 4, 5)$ and is parallel to the vectors $\overrightarrow{AB} = \langle -1, -4, -2\rangle$ and $\overrightarrow{AC} = \langle 3, 4, -2\rangle$. A normal vector can be computed as

$$\overrightarrow{AB}\times\overrightarrow{AC} = \langle 16, -8, 8\rangle.$$

Since it is 8 times $\langle 2, -1, 1\rangle$, one might just take this as a normal vector. Then the plane is of the form $2x-y+z = a$ where $2\cdot 2-4+5 = a$. So $a = 5$, and the plane has an equation $2x-y+z = 5$.

(4) This is the same as the plane that passes through $(2, 0, -4)$ and is parallel to the vectors $\overrightarrow{AB} = \langle -2, 1, 8\rangle$ and $\overrightarrow{AC} = \langle 3, 2, 9\rangle$. A normal vector can be computed as

$$\overrightarrow{AB}\times\overrightarrow{AC} = \langle -7, 42, -7\rangle.$$

Since this is 7 times $\langle -1, 6, -1\rangle$, one might just take this as a normal vector. Then the plane is of the form $-x+6y-z = a$, where $-2+6\cdot 0-(-4) = a$. So $a = 2$, and the plane has an equation $-x+6y-z = 2$.

□

Exercise 3. Determine whether the given objects intersect or not. Determine whether the given objects are parallel or not.

(1) The line $L_1 : x = t, y = 6-2t, z = -7+t$ and the plane $P_1 : 2x+3y+z = -1$.

(2) The line L_1 passing through $A = (2, -1, 5)$ and $B = (0, -7, 9)$, and the plane $P_1 : 5x-3x-2z = 2$.

(3) The plane $P_1 : 2x+5y+z = 3$ and the plane $P_2 : 6x+15y+3z = 2$.

(4) The plane $P_1 : -x+2y+3z = 0$ and the plane $P_2 : 3x+2y-z = 3$.

Solution.

(1) Plug the parametric equation for L_1 to the equation for P_1, and we get

$$2t+3(6-2t)+(-7+t) = -1,$$

or

$$2t+18-6t-7+t = -1,$$

or

$$12 = 3t,$$

or $t = 4$. So L_1 and P_1 intersect.

(2) The direction of the line is $\overrightarrow{AB} = \langle -2, -6, 4\rangle$. Since $\langle -2, -6, 4\rangle\cdot\langle 5, -3, -2\rangle = (-2)\cdot 5+(-6)\cdot(-3)+4\cdot(-2) = -10+18-8 = 0$, they are parallel.

(3) Since $\langle 2, 5, 1\rangle$ and $\langle 6, 15, 3\rangle$ are parallel, the two planes are parallel.

(4) Since $\langle -1, 2, 3\rangle$ and $\langle 3, 2, -1\rangle$ are not parallel, the two planes intersect.

□

Exercise 4. Find an equation of the following lines.

(1) The intersection between the planes $P_1 : 3x-y+2z = 2$ and $P_2 : -2x+y+z = 0$.

(2) The intersection between the plane $P_1 : 7x-3y+4z = -7$ and the plane P_2 that passes through $(4, 4, 0)$ and is orthogonal to the vector $\vec{n} = \langle 1, -1, 1\rangle$.

Solution.

(1) The line has a direction $\langle 3,-1,2\rangle \times \langle -2,1,1\rangle = \langle -3,-7,1\rangle$. To find a point in common let's say we set $z=0$. Then we need to solve

$$3x-y=2, \quad -2x+y=0.$$

You add them, and get $x=2$, and $y=4$. So it is the line that passes through $(2,4,0)$ and is parallel to $\langle -3,-7,1\rangle$. So it can be given as

$$x=2-3t, y=4-7t, z=t.$$

(2) The line has a direction $\langle 7,-3,4\rangle \times \angle 1,-1,1\rangle = \langle 1,-3,-4\rangle$. The equation for P_2 can be given by $x-y+z=a$, and if you plug $(4,4,0)$ you get $0=a$, so it is just $x-y+z=0$. To find a point in common, let's say $x=0$. Then we need to solve

$$-3y+4z=-7, \quad -y+z=0.$$

So $y=z=-7$. So it is the line that passes through $(0,-7,-7)$ and is parallel to $\langle 1,-3,-4\rangle$. So it can be given as

$$x=t, y=--7-3t, z=-7-4t.$$

□

Exercise 5. Find the angle between the following objects.

(1) The line $L_1 : x=t, y=2-2t, z=6+2t$ and the plane $P_1 : x-y=0$.
(2) The line $L_1 : x=94+t, y=e^{150\pi}-t, z=0.01524$ and the plane

$$P_1 : (1+3\sqrt{3})x+(1-3\sqrt{3})y-4z=e^{2022^{e^{2022}}}.$$

(3) The plane P_1 that passes through $A=(1,-1,1)$, $B=(\frac{1}{2},2,2)$ and $C=(2,5,7)$ and the plane $P_2 : (2+6\sqrt{3})x+(3+2\sqrt{3})y+(6-3\sqrt{3})z=0$.

Solution.

(1) If the angle is θ, we have

$$\sin\theta = \frac{\langle 1,-2,2\rangle \cdot \langle 1,-1,0\rangle}{|\langle 1,-2,2\rangle|\,|\langle 1,-1,0\rangle|} = \frac{1+2}{\sqrt{1+4+4}\sqrt{1+1}} = \frac{3}{\sqrt{9}\sqrt{2}} = \frac{1}{\sqrt{2}},$$

so $\theta = \frac{\pi}{4}$.
(2) If the angle is θ, we have

$$\sin\theta = \frac{\langle 1,-1,0\rangle \cdot \langle 1+3\sqrt{3}, 1-3\sqrt{3}, -4\rangle}{|\langle 1,-1,0\rangle|\,|\langle 1+3\sqrt{3}, 1-3\sqrt{3}, -4\rangle|} = \frac{(1+3\sqrt{3})-(1-3\sqrt{3})}{\sqrt{1+1}\sqrt{(1+3\sqrt{3})^2+(1-3\sqrt{3})^2+16}}$$

$$= \frac{6\sqrt{3}}{\sqrt{2}\sqrt{(1+6\sqrt{3}+27)+(1-6\sqrt{3}+27)+16}} = \frac{6\sqrt{3}}{\sqrt{2}\sqrt{28+28+16}} = \frac{6\sqrt{3}}{\sqrt{144}} = \frac{\sqrt{3}}{2}.$$

So $\theta = \frac{\pi}{3}$.

(3) First we need to find a normal vector of P_1. It can be given as $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -\frac{1}{2}, 3, 1\rangle \times \langle 1, 6, 6\rangle = \langle 12, 4, -6\rangle$. Since this is twice $\langle 6, 2, -3\rangle$, we can take this as a normal vector for simplicity. If the angle is θ, then we have

$$\cos\theta = \frac{\langle 6, 2, -3\rangle \cdot \langle 2+6\sqrt{3}, 3+2\sqrt{3}, 6-3\sqrt{3}\rangle}{|\langle 6, 2, -3\rangle|\,|\langle 2+6\sqrt{3}, 3+2\sqrt{3}, 6-3\sqrt{3}\rangle|}.$$

We have

$$\langle 6, 2, -3\rangle \cdot \langle 2+6\sqrt{3}, 3+2\sqrt{3}, 6-3\sqrt{3}\rangle = 6(2+6\sqrt{3}) + 2(3+2\sqrt{3}) - 3(6-3\sqrt{3}) = 49\sqrt{3},$$

and

$$|\langle 6, 2, -3\rangle| = \sqrt{36+4+9} = \sqrt{49} = 7,$$

and

$$|\langle 2+6\sqrt{3}, 3+2\sqrt{3}, 6-3\sqrt{3}\rangle| = \sqrt{(2+6\sqrt{3})^2 + (3+2\sqrt{3})^2 + (6-3\sqrt{3})^2}$$

$$= \sqrt{(4+24\sqrt{3}+108) + (9+12\sqrt{3}+12) + (36-36\sqrt{3}+27)} = \sqrt{196} = 14,$$

so

$$\cos\theta = \frac{49\sqrt{3}}{7\cdot 14} = \frac{\sqrt{3}}{2},$$

so $\theta = \frac{\pi}{6}$.

□

7. Lines and planes, continued

Exercise 1. Find the distance between the given objects.

(1) The point $A = (3, -1, 0)$ and the line $L_1 : x = -1 - t, y = 4 - 2t, z = -3 + 2t$.
(2) The point $A = (-2, 1, 2)$ and the plane $P_1 : -6x + 3y + 6z = -2$.
(3) The line $L_1 : x = 2 - 3t, y = 2t, z = 3 - 2t$ and the plane $P_1 : 2x - 6y - 9z = 3$.
(4) The line $L_1 : x = -18 + 3t, y = -4 + 2t, z = -11 + t$ and the plane $P_1 : x + 2y + z = 3$.
(5) The line $L_1 : x = 5 + t, y = -2 + t, z = 6t$ and the line $L_2 : x = 7 - s, y = -s, z = 5 - 6s$.
(6) The line $L_1 : x = 3 + 2t, y = 1 + t, z = 5 - t$ and the line $L_2 : x = 5 - s, y = 2 - 2s, z = 3 - 4s$.
(7) The plane $P_1 : -x + 3y - 4z = 11$ and the plane $P_2 : 11x + 4y - 3z = 0$
(8) The plane $P_1 : 3x - 2y + 2z = -5$ and the plane P_2 that passes through $A = (0, 3, 1)$, $B = (-2, 0, 1)$ and $C = (4, 8, 0)$.

Solution.

(1) Let $B = (-1, 4, -3)$. Then the distance is the length of the vector $\overrightarrow{BA} - \operatorname{proj}_{\langle -1,-2,2\rangle} \overrightarrow{BA}$. Note $\overrightarrow{BA} = \langle 4, -5, 3\rangle$. So

$$\operatorname{proj}_{\langle -1,-2,2\rangle} \overrightarrow{BA} = \frac{\langle -1, -2, 2\rangle \cdot \langle 4, -5, 3\rangle}{|\langle -1, -2, 2\rangle|^2}\langle -1, -2, 2\rangle$$

$$= \frac{-4+10+6}{9}\langle -1, -2, 2\rangle = \frac{12}{9}\langle -1, -2, 2\rangle = \langle -\frac{4}{3}, -\frac{8}{3}, \frac{8}{3}\rangle.$$

So

$$\overrightarrow{BA} - \operatorname{proj}_{\langle -1,-2,2\rangle} \overrightarrow{BA} = \langle \frac{16}{3}, -\frac{7}{3}, \frac{1}{3}\rangle.$$

So the distance is the length of this vector, which is $\frac{\sqrt{256+49+1}}{3} = \frac{\sqrt{306}}{3} = \sqrt{34}$.

(2) We use the formula

$$\frac{|-6\cdot(-2)+3\cdot 1+6\cdot 2+2|}{\sqrt{(-6)^2+3^2+6^2}} = \frac{|12+3+12+2|}{\sqrt{36+9+36}} = \frac{29}{9}.$$

(3) Firstly we check they are parallel, because the direction of L_1, $\langle -3,2,-2\rangle$, is orthogonal to the normal vector of P_1, $\langle 2,-6,-9\rangle$, as

$$\langle -3,2,-2\rangle \cdot \langle 2,-6,-9\rangle = 0.$$

So we pick a point of L_1, say $(2,0,3)$, and use the formula for the distance between a point and a plane. So the distance is

$$\frac{|2\cdot 2-6\cdot 0-9\cdot 3-3|}{\sqrt{2^2+(-6)^2+(-9)^2}} = \frac{|4-27-3|}{\sqrt{4+36+81}} = \frac{26}{11}.$$

(4) It intersects at $(-3,6,-6)$, so the distance is 0.

(5) The two lines are obviously parallel, so this is the same as the distance between $A = (5,-2,0)$ (a point on L_1) and the line L_2.

We do as we did in (1). Pick a point on the line L_2, say $B = (7,0,5)$. The direction vector of L_2 can be taken as $\vec{v} = \langle -1,-1,-6\rangle$. Then, the distance between A and the line L_2 would be the distance of the vector

$$\overrightarrow{BA} - \mathrm{proj}_{\vec{v}}\,\overrightarrow{BA} = \langle 2,2,5\rangle - \frac{\langle 2,2,5\rangle\cdot\langle -1,-1,-6\rangle}{|\langle -1,-1,-6\rangle|^2}\langle -1,-1,-6\rangle$$

$$= \langle 2,2,5\rangle - \frac{-34}{38}\langle -1,-1,-6\rangle = \langle 2-\frac{17}{19}, 2-\frac{17}{19}, 5-\frac{102}{19}\rangle = \langle \frac{21}{19}, \frac{21}{19}, -\frac{7}{19}\rangle.$$

So the distance is

$$\frac{\sqrt{21^2+21^2+7^2}}{19} = \frac{\sqrt{931}}{19} = \frac{\sqrt{7^2\cdot 19}}{19} = \frac{7}{\sqrt{19}}.$$

(6) We first see if the two lines are intersecting, parallel or skew. As the direction vetcors $\langle 2,1,-1\rangle$ and $\langle -1,-2,-4\rangle$ are not parallel, the lines are definitely not parallel. If they intersect, there are values of s, t such that

$$3+2t = 5-s, \quad 1+t = 2-2s, \quad 5-t = 3-4s,$$

are satisfied. Using the first two equations, we see that

$$1 = (3+2t)-2(1+t) = (5-s)-2(2-2s) = 1+3s,$$

so $s=0$ and $t=1$. Then the third equation is $4=3$, which is a contradiction. So they do not intersect, and therefore are skew.

To get the distance between two lines skew to each other, we pick a plane P_1 that contains L_1 and is parallel to L_2, and compute the distance between P_1 and L_2. The plane P_1 passes through a point $(3,1,5)$ and is orthogonal to $\langle 2,1,-1\rangle \times \langle -1,-2,-4\rangle = \langle -6,9,-3\rangle$. We can actually take a normal vector to be $\langle -2,3,-1\rangle$, so the plane P_1 should have the equation $-2x+3y-z = a$ where $-2\cdot 3+3\cdot 1-5 = a$. So $a = -6+3-5 = -8$, and therefore the plane P_1 has the equation $-2x+3y-z+8=0$. Pick a point in L_2, say $(5,2,3)$, and the distance between P_1 and L_2 is the distance between P_1 and this point, so

$$\frac{|-2\cdot 5+3\cdot 2-3+8|}{\sqrt{(-2)^2+3^2+(-1)^2}}=\frac{|-10+6-3+8|}{\sqrt{14}}=\frac{1}{\sqrt{14}}.$$

(7) The two planes are not parallel, so they intersect, and the distance is 0.

(8) We first write P_2 in an equation form, so we need to find a normal vector. This can be computed as $\overrightarrow{AB}\times\overrightarrow{AC}=\langle -2,-3,0\rangle\times\langle 4,5,-1\rangle=\langle 3,-2,2\rangle$. So P_1 and P_2 are evidently parallel. So the distance between P_1 and P_2 is the same as the distance between P_1 and a point on P_2, say $A=(0,3,1)$. So we can use the formula

$$\frac{|3\cdot 0-2\cdot 3+2\cdot 1+5|}{\sqrt{3^2+(-2)^2+2^2}}=\frac{|-6+2+5|}{\sqrt{9+4+4}}=\frac{1}{\sqrt{17}}.$$

□